7.5: Periodic and Piecewise Continuous Input Functions

Theorem 1. (Translation of the t-axis)
If $\mathcal{L}\{f(t)\}$ exists for $s > c$, then

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

and

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

for $s > c + a$ where $u(t-a) = u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$.

Example 1. Find $\mathcal{L}^{-1}\{\frac{e^{-as}}{s^3}\}$.

Example 2. Find $\mathcal{L}\{g(t)\}$ if

$$g(t) = \begin{cases} 0 & \text{if } t < 3 \\ t^2 & \text{if } t \geq 3. \end{cases}$$

Example 3. Find $\mathcal{L}\{f(t)\}$ if

$$f(t) = \begin{cases} \cos 2t & \text{if } 0 \leq t < 2\pi \\ 0 & \text{if } t \geq 2\pi. \end{cases}$$

Example 4. Consider the RLC circuit $R = 110\Omega$, $L = 1H$, $C = 0.001F$ and a battery supplying $E_0 = 90V$. Initially there is no current in the circuit and no charge on the capacitor. At time $t = 0$ the switch is closed and left closed for 1 second. At time $t = 1$ it is opened and left open thereafter. Find the resulting current in the circuit if the equation is given by

$$\frac{di}{dt} + 110i + 1000q = e(t).$$

Exercise 1. A mass that weighs 32 lb is attached to the free end of a long, light spring that is stretched 1ft by a force of 4 lb. The mass is initially at rest in its equilibrium position. Beginning at time $t = 0$ (seconds), an external force $f(t) = \cos 2t$ is applied to the mass, but at time $t = 2\pi$ this force is turned off (abruptly discontinued) and the mass is allowed to continue its motion unimpeded. Find the resulting position function $x(t)$ of the mass is the resulting differential equation is

$$x'' + 4x = f(t); \quad x(0) = x'(0) = 0.$$

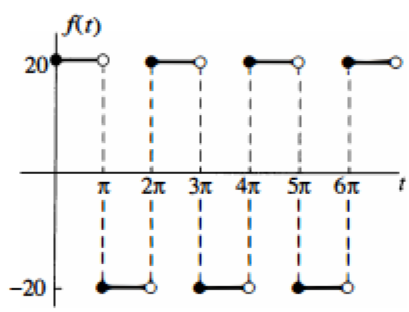
Theorem 2. (Transforms of Periodic Functions)
Let $f(t)$ be periodic with period p and piecewise continuous for $t \geq 0$. Then the transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > 0$ and is given by

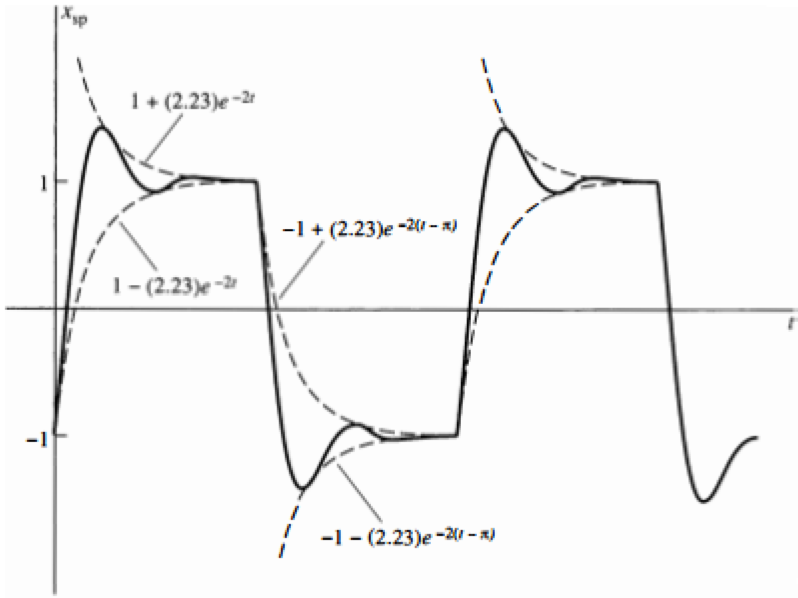
$$F(s) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-}stf(t)dt.$$

Example 5. Let $f(t) = (-1)^{\lceil t/a \rceil}$ be the square-wave function of period $p = 2a$. Find $\mathcal{L}\{f(t)\}$.

Example 6. Consider a mass-spring-dashpot system with $m = 1$, $c = 4$, and $k = 20$ in appropriate units. Suppose that the system is initially at rest at equilibrium and that the mass is acted on beginning at time $t = 0$ by the external force $f(t)$ which is the square wave function with amplitude 20 and period 2π. Find the position function $x(t)$ is the associated differential equation is

$$x'' + 4x' + 20x = f(t); \quad x(0) = x'(0) = 0.$$





Homework. 1-29, 33-37 (odd)